Direct Optimization of Frame-to-Frame Rotation

Laurent Kneip and Simon Lynen, ICCV 2013
Supplemental derivations

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1 Closed-form solution of $\lambda_{\mathbf{M},min}$

As mentioned in Section 3.1, the easiest way to solve the problem

$$\mathbf{R} = \text{argmin}_{\mathbf{R}} \lambda_{\mathbf{M},min}$$

consists of applying the gradient descent approach. More interestingly, $\lambda_{\mathbf{M},min}$ can be solved in closed form. Since $\mathbf{M}$ is a real symmetric positive-definite matrix, the eigenvalues of $\mathbf{M}$ are real positive values. They are given by the roots of the third-order polynomial $det(\mathbf{M} - \lambda \mathbf{I}_3) = 0$, which can be solved in closed form. Let $\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0$ be this constraint. The smallest root is then always given by

$$\begin{aligned}
b_1 &= -m_{11} - m_{22} - m_{33} \\
b_2 &= -m_{13}^2 - m_{23}^2 - m_{12}^2 + m_{11}m_{22} + m_{11}m_{33} + m_{22}m_{33} \\
b_3 &= m_{22}m_{13}^2 + m_{11}m_{23}^2 + m_{33}m_{12}^2 - m_{11}m_{22}m_{33} - 2m_{12}m_{23}m_{13} \\
s &= 2b_1^3 - 9b_1b_2 + 27b_3 \\
t &= 4(b_1^2 - 3b_2)^3 \\
k &= \left(\frac{\sqrt{t}}{2}\right)^{\frac{1}{3}} \cos\left(\frac{\arccos\left(\frac{s}{\sqrt{t}}\right)}{3}\right) \\
\Rightarrow \lambda_{\mathbf{M},\text{min}} &= \min\left[\text{eig}\left(\mathbf{M}\right)\right] = \frac{-b_1 - 2k}{3}
\end{aligned}$$

Gradient descent is easily realized via a numerical computation of the Jacobians. However, as explained in Section 3.1, the application of Levenberg-Marquardt requires the availability of more constraints and notably constraints to zero. These are given by all partial derivatives w.r.t. the Cayley parameters set to zero, which constitutes the definition of a minimum. They are given by

$$\begin{cases} \frac{\partial \lambda_{\mathbf{M},min}}{\partial x} = 0 \\ \frac{\partial \lambda_{\mathbf{M},min}}{\partial y} = 0 \\ \frac{\partial \lambda_{\mathbf{M},min}}{\partial z} = 0 \end{cases}$$

The partial derivatives can again be computed in closed form. As an example, $\frac{\partial \lambda_{\mathbf{M},min}}{\partial x}$ is given by

$$\begin{aligned}
\frac{\partial \lambda_{\mathbf{M},min}}{\partial x} &= \frac{-\frac{\partial b_1}{\partial x} - 2\frac{\partial k}{\partial x}}{3} \\
\frac{\partial k}{\partial x} &= \frac{1}{3}\left(\frac{\sqrt{t}}{2}\right)^{-\frac{2}{3}} \frac{t^{-\frac{1}{2}}}{4}\frac{\partial t}{\partial x}\cos\left(\frac{\arccos\left(\frac{s}{\sqrt{t}}\right)}{3}\right) \\
&\quad + \left(\frac{\sqrt{t}}{2}\right)^{\frac{1}{3}} \sin\left(\frac{\arccos\left(\frac{s}{\sqrt{t}}\right)}{3}\right) \frac{\frac{\partial s}{\partial x}\sqrt{t} - \frac{1}{2}s \cdot t^{-\frac{1}{2}}\frac{\partial t}{\partial x}}{3t\sqrt{1-\frac{s^2}{t}}} \\
\frac{\partial s}{\partial x} &= 6b_1^2\frac{\partial b_1}{\partial x} - 9\frac{\partial b_1}{\partial x}b_2 - 9b_1\frac{\partial b_2}{\partial x} + 27\frac{\partial b_3}{\partial x} \\
\frac{\partial t}{\partial x} &= 12(b_1^2 - 3b_2)^2(2b_1\frac{\partial b_1}{\partial x} - 3\frac{b_2}{\partial x}) \\
\frac{\partial b_1}{\partial x} &= -\frac{\partial m_{11}}{\partial x} - \frac{\partial m_{22}}{\partial x} - \frac{\partial m_{33}}{\partial x} \\
\frac{\partial b_2}{\partial x} &= -2m_{13}\frac{\partial m_{13}}{\partial x} - 2m_{23}\frac{\partial m_{23}}{\partial x} - 2m_{12}\frac{\partial m_{12}}{\partial x} \\
&\quad + \frac{\partial m_{11}}{\partial x}m_{22} + m_{11}\frac{\partial m_{22}}{\partial x} + \frac{\partial m_{11}}{\partial x}m_{33} \\
&\quad + m_{11}\frac{\partial m_{33}}{\partial x} + \frac{\partial m_{22}}{\partial x}m_{33} + m_{22}\frac{\partial m_{33}}{\partial x} \\
\frac{\partial b_3}{\partial x} &= \frac{\partial m_{22}}{\partial x}m_{13}^2 + 2m_{22}m_{13}\frac{\partial m_{13}}{\partial x} + \frac{\partial m_{11}}{\partial x}m_{23}^2 \\
&\quad + 2m_{11}m_{23}\frac{\partial m_{23}}{\partial x} + \frac{\partial m_{33}}{\partial x}m_{12}^2 + 2m_{33}m_{12}\frac{\partial m_{12}}{\partial x} \\
&\quad - \frac{\partial m_{11}}{\partial x}m_{22}m_{33} - m_{11}\frac{\partial m_{22}}{\partial x}m_{33} - m_{11}m_{22}\frac{\partial m_{33}}{\partial x} \\
&\quad - 2(\frac{\partial m_{12}}{\partial x}m_{23}m_{13} + m_{12}\frac{\partial m_{23}}{\partial x}m_{13} + m_{12}m_{23}\frac{\partial m_{13}}{\partial x}) \\
\frac{\partial m_{11}}{\partial x} &= 2\frac{\partial \mathbf{r}_3}{\partial x}(\sum_{i=1}^{n} f_{yi}^2\mathbf{F}'_i)\mathbf{r}_3^T - 2\frac{\partial \mathbf{r}_3}{\partial x}(\sum_{i=1}^{n} f_{yi}f_{zi}\mathbf{F}'_i)\mathbf{r}_2^T \\
&\quad - 2\mathbf{r}_3(2\sum_{i=1}^{n} f_{yi}f_{zi}\mathbf{F}'_i)\left(\frac{\partial \mathbf{r}_2}{\partial x}\right)^T + 2\frac{\partial \mathbf{r}_2}{\partial x}(\sum_{i=1}^{n} f_{zi}^2\mathbf{F}'_i)\mathbf{r}_2^T
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial m_{12}}{\partial x} &= \frac{\partial \mathbf{r}_3}{\partial x}(\sum_{i=1}^{n} f_{yi}f_{zi}\mathbf{F}'_i)\mathbf{r}_1^T + \mathbf{r}_3(\sum_{i=1}^{n} f_{yi}f_{zi}\mathbf{F}'_i)\left(\frac{\partial \mathbf{r}_1}{\partial x}\right)^T \\
&\quad -2\frac{\partial \mathbf{r}_3}{\partial x}(\sum_{i=1}^{n} f_{xi}f_{yi}\mathbf{F}'_i)\mathbf{r}_3^T - \frac{\partial \mathbf{r}_2}{\partial x}(\sum_{i=1}^{n} f_{zi}^2\mathbf{F}'_i)\mathbf{r}_1^T \\
&\quad -\mathbf{r}_2(\sum_{i=1}^{n} f_{zi}^2\mathbf{F}'_i)\left(\frac{\partial \mathbf{r}_1}{\partial x}\right)^T + \frac{\partial \mathbf{r}_2}{\partial x}(\sum_{i=1}^{n} f_{xi}f_{zi}\mathbf{F}'_i)\mathbf{r}_3^T \\
&\quad +\mathbf{r}_2(\sum_{i=1}^{n} f_{xi}f_{zi}\mathbf{F}'_i)\left(\frac{\partial \mathbf{r}_3}{\partial x}\right)^T \\
\frac{\partial m_{13}}{\partial x} &= \frac{\partial \mathbf{r}_3}{\partial x}(\sum_{i=1}^{n} f_{xi}f_{yi}\mathbf{F}'_i)\mathbf{r}_2^T + \mathbf{r}_3(\sum_{i=1}^{n} f_{xi}f_{yi}\mathbf{F}'_i)\left(\frac{\partial \mathbf{r}_2}{\partial x}\right)^T \\
&\quad -\frac{\partial \mathbf{r}_3}{\partial x}(\sum_{i=1}^{n} f_{yi}^2\mathbf{F}'_i)\mathbf{r}_1^T - \mathbf{r}_3(\sum_{i=1}^{n} f_{yi}^2\mathbf{F}'_i)\left(\frac{\partial \mathbf{r}_1}{\partial x}\right)^T \\
&\quad -2\frac{\partial \mathbf{r}_2}{\partial x}(\sum_{i=1}^{n} f_{xi}f_{zi}\mathbf{F}'_i)\mathbf{r}_2^T + \frac{\partial \mathbf{r}_2}{\partial x}(\sum_{i=1}^{n} f_{yi}f_{zi}\mathbf{F}'_i)\mathbf{r}_1^T \\
&\quad +\mathbf{r}_2(\sum_{i=1}^{n} f_{yi}f_{zi}\mathbf{F}'_i)\left(\frac{\partial \mathbf{r}_1}{\partial x}\right)^T \\
\frac{\partial m_{22}}{\partial x} &= 2\frac{\partial \mathbf{r}_1}{\partial x}(\sum_{i=1}^{n} f_{zi}^2\mathbf{F}'_i)\mathbf{r}_1^T - 2\frac{\partial \mathbf{r}_1}{\partial x}(\sum_{i=1}^{n} f_{xi}f_{zi}\mathbf{F}'_i)\mathbf{r}_3^T \\
&\quad -2\mathbf{r}_1(\sum_{i=1}^{n} f_{xi}f_{zi}\mathbf{F}'_i)\left(\frac{\partial \mathbf{r}_3}{\partial x}\right)^T + 2\frac{\partial \mathbf{r}_3}{\partial x}(\sum_{i=1}^{n} f_{xi}^2\mathbf{F}'_i)\mathbf{r}_3^T \\
\frac{\partial m_{23}}{\partial x} &= \frac{\partial \mathbf{r}_1}{\partial x}(\sum_{i=1}^{n} f_{xi}f_{zi}\mathbf{F}'_i)\mathbf{r}'_2 + \mathbf{r}_1(\sum_{i=1}^{n} f_{xi}f_{zi}\mathbf{F}'_i)\left(\frac{\partial \mathbf{r}_2}{\partial x}\right)^T \\
&\quad -2\frac{\partial \mathbf{r}_1}{\partial x}(\sum_{i=1}^{n} f_{yi}f_{zi}\mathbf{F}'_i)\mathbf{r}_1^T - \frac{\partial \mathbf{r}_3}{\partial x}(\sum_{i=1}^{n} f_{xi}^2\mathbf{F}'_i)\mathbf{r}_2^T \\
&\quad -\mathbf{r}_3(\sum_{i=1}^{n} f_{xi}^2\mathbf{F}'_i)\left(\frac{\partial \mathbf{r}_2}{\partial x}\right)^T + \frac{\partial \mathbf{r}_3}{\partial x}(\sum_{i=1}^{n} f_{xi}f_{yi}\mathbf{F}'_i)\mathbf{r}_1^T \\
&\quad +\mathbf{r}_3(\sum_{i=1}^{n} f_{xi}f_{yi}\mathbf{F}'_i)\left(\frac{\partial \mathbf{r}_1}{\partial x}\right)^T \\
\frac{\partial m_{33}}{\partial x} &= 2\frac{\partial \mathbf{r}_2}{\partial x}(\sum_{i=1}^{n} f_{xi}^2\mathbf{F}'_i)\mathbf{r}_2^T - 2\frac{\partial \mathbf{r}_1}{\partial x}(\sum_{i=1}^{n} f_{xi}f_{yi}\mathbf{F}'_i)\mathbf{r}_2^T \\
&\quad -2\mathbf{r}_1(\sum_{i=1}^{n} f_{xi}f_{yi}\mathbf{F}'_i)\left(\frac{\partial \mathbf{r}_2}{\partial x}\right)^T + 2\frac{\partial \mathbf{r}_1}{\partial x}(\sum_{i=1}^{n} f_{yi}^2\mathbf{F}'_i)\mathbf{r}_1^T
\end{aligned}
$$

$$\frac{\partial \mathbf{R}}{\partial x} = \begin{pmatrix} \frac{\partial \mathbf{r}_1}{\partial x} \\ \frac{\partial \mathbf{r}_2}{\partial x} \\ \frac{\partial \mathbf{r}_3}{\partial x} \end{pmatrix} = \begin{pmatrix} 2x & 2y & 2z \\ 2y & -2x & -2 \\ 2z & 2 & -2x \end{pmatrix}$$

The partial derivatives with respect to other Cayley parameters (i.e. y and z) are essentially the same, except for the final rotation matrix derivatives. These result to

$$\frac{\partial \mathbf{R}}{\partial y} = \begin{pmatrix} \frac{\partial \mathbf{r}_1}{\partial y} \\ \frac{\partial \mathbf{r}_2}{\partial y} \\ \frac{\partial \mathbf{r}_3}{\partial y} \end{pmatrix} = \begin{pmatrix} -2y & 2x & 2 \\ 2x & 2y & 2z \\ -2 & 2z & -2y \end{pmatrix}$$

and

$$\frac{\partial \mathbf{R}}{\partial z} = \begin{pmatrix} \frac{\partial \mathbf{r}_1}{\partial z} \\ \frac{\partial \mathbf{r}_2}{\partial z} \\ \frac{\partial \mathbf{r}_3}{\partial z} \end{pmatrix} = \begin{pmatrix} -2z & -2 & 2x \\ 2 & -2z & 2y \\ 2x & 2y & 2z \end{pmatrix}$$

2 Derivation of bound on variation of smallest eigenvalue

2.1 Bound on the variation of R

We have seen in Section 3.2 that

$$\mathbf{R} = \begin{pmatrix} 1+x^2-y^2-z^2 & 2(-z+xy) & 2(y+xz) \\ 2(z+xy) & 1-x^2+y^2-z^2 & 2(-x+yz) \\ 2(-y+xz) & 2(x+yz) & 1-x^2-y^2+z^2 \end{pmatrix}$$

represents a good parametrization of the rotation. Now let $x' = x+x^*$, $y' = y+y^*$, and $z' = z+z^*$ be varied Cayley parameters with bounds on the absolute variation given by $|x^*|, |y^*|, |z^*| \leq \epsilon$. Let $\mathbf{R}'$ be the rotation matrix that corresponds to x', y', and z', and $\mathcal{R}^*$ be the concrete variation of the rotation matrix such that $\mathbf{R}' = \mathbf{R} + \mathcal{R}^*$. The absolute bound $\mathcal{R}$ on the variation of the rotation matrix is derived as follows.

$$\begin{aligned}
|\mathcal{R}^*| &= |\mathbf{R}' - \mathbf{R}| \\
&= \left| \begin{pmatrix} 1+x'^2-y'^2-z'^2 & 2(-z'+x'y') & 2(y'+x'z') \\ 2(z'+x'y') & 1-x'^2+y'^2-z'^2 & 2(-x'+y'z') \\ 2(-y'+x'z') & 2(x'+y'z') & 1-x'^2-y'^2+z'^2 \end{pmatrix} - \right. \\
&\quad \left. \begin{pmatrix} 1+x^2-y^2-z^2 & 2(-z+xy) & 2(y+xz) \\ 2(z+xy) & 1-x^2+y^2-z^2 & 2(-x+yz) \\ 2(-y+xz) & 2(x+yz) & 1-x^2-y^2+z^2 \end{pmatrix} \right|
\end{aligned}$$

$$= \begin{pmatrix} |2xx^*+x^{*2}-2yy^*-y^{*2}-2zz^*-z^{*2}| & 2|xy^*+yx^*+x^*y^*-z^*| & .. \\ 2|xy^*+yx^*+x^*y^*+z^*| & |-2xx^*-x^{*2}+2yy^*+y^{*2}-2zz^*-z^{*2}| & .. \\ 2|xz^*+zx^*+x^*z^*-y^*| & 2|yz^*+zy^*+y^*z^*+x^*| & .. \end{pmatrix}$$

$$\begin{pmatrix} .. & 2|xz^*+zx^*+x^*z^*+y^*| \\ .. & 2|yz^*+zy^*+y^*z^*-x^*| \\ .. & |-2xx^*-x^{*2}-2yy^*-y^{*2}+2zz^*+z^{*2}| \end{pmatrix}$$

$$\leq \begin{pmatrix} 2(|x|+|y|+|z|)\epsilon+3\epsilon^2 & 2(|x|+|y|+1)\epsilon+\epsilon^2 & 2(|x|+|z|+1)\epsilon+\epsilon^2 \\ 2(|x|+|y|+1)\epsilon+\epsilon^2 & 2(|x|+|y|+|z|)\epsilon+3\epsilon^2 & 2(|y|+|z|+1)\epsilon+\epsilon^2 \\ 2(|x|+|z|+1)\epsilon+\epsilon^2 & 2(|y|+|z|+1)\epsilon+\epsilon^2 & 2(|x|+|y|+|z|)\epsilon+3\epsilon^2 \end{pmatrix}$$

$$\Rightarrow \mathcal{R} = \begin{pmatrix} 2(|x|+|y|+|z|)\epsilon+3\epsilon^2 & 2(|x|+|y|+1)\epsilon+2\epsilon^2 & 2(|x|+|z|+1)\epsilon+2\epsilon^2 \\ 2(|x|+|y|+1)\epsilon+2\epsilon^2 & 2(|x|+|y|+|z|)\epsilon+3\epsilon^2 & 2(|y|+|z|+1)\epsilon+2\epsilon^2 \\ 2(|x|+|z|+1)\epsilon+2\epsilon^2 & 2(|y|+|z|+1)\epsilon+2\epsilon^2 & 2(|x|+|y|+|z|)\epsilon+3\epsilon^2 \end{pmatrix}$$

2.2 Bound on the variation of M

As indicated in (4), the elements of M are a function of the rows of the rotation matrix $\mathbf{R}$. By having a bound on the variation of the elements of the rotation matrix $\mathbf{R}$—namely $\mathcal{R}$—we can now derive a bound on the variation of the elements of $\mathbf{M}$, denoted by $\mathcal{M}$.

To start with, let's simplify the problem a bit by noting that all the elements m of $\mathbf{M}$ are linear combinations of terms of the form $\mathbf{r}_a \mathbf{J} \mathbf{r}_b^T$, with $a, b \in \{1, 2, 3\}$ and

$\mathbf{J}$ being the symmetric summation term over the bearing vector correspondences. Denoting this function by $q_{ab}(\mathbf{J})$, we have

$$\begin{aligned} q_{ab}(\mathbf{J}) = \mathbf{r}_a \mathbf{J} \mathbf{r}_b^T &= r_{a1}r_{b1}j_{11} + r_{a2}r_{b1}j_{12} + r_{a3}r_{b1}j_{13} \\ &+ r_{a1}r_{b2}j_{12} + r_{a2}r_{b2}j_{22} + r_{a3}r_{b2}j_{23} \\ &+ r_{a1}r_{b3}j_{13} + r_{a2}r_{b3}j_{23} + r_{a3}r_{b3}j_{33}, \end{aligned}$$

where r_{ij} denotes an element of $\mathbf{R}$ and j_{ij} an element of $\mathbf{J}$. Let's denote by $q^{'}_{ab}(\mathbf{J})$ the result of this function based on a varied input rotation $\mathbf{R}'$. Absolutely bounding the concrete variation $q^*_{ab}(\mathbf{J}) = q^{'}_{ab}(\mathbf{J}) - q_{ab}(\mathbf{J})$ based on the maximum absolute rotation variation $\mathcal{R}$ then results in a bound on the absolute variation of $q_{ab}(\mathbf{J})$, denoted by $\theta_{ab}(\mathbf{J})$. If $r^{'}_{ij}$ denotes the elements of $\mathbf{R}'$ and $\mathcal{R}_{ij}$ the elements of $\mathcal{R}$, the derivation looks as follows.

$$\begin{aligned} |q^*_{ab}(\mathbf{J})| &= |q^{'}_{ab}(\mathbf{J}) - q_{ab}(\mathbf{J})| \\ &= |(r_{a1} + r^{'}_{a1})(r_{b1} + r^{'}_{b1})j_{11} - r_{a1}r_{b1}j_{11} + (r_{a2} + r^{'}_{a2})(r_{b1} + r^{'}_{b1})j_{11} \\ &\quad -r_{a2}r_{b1}j_{12} + (r_{a3} + r^{'}_{a3})(r_{b1} + r^{'}_{b1})j_{13} - r_{a3}r_{b1}j_{13} \\ &\quad +(r_{a1} + r^{'}_{a1})(r_{b2} + r^{'}_{b2})j_{12} - r_{a1}r_{b2}j_{12} \\ &\quad +(r_{a2} + r^{'}_{a2})(r_{b2} + r^{'}_{b2})j_{22} - r_{a2}r_{b2}j_{22} + (r_{a3} + r^{'}_{a3})(r_{b2} + r^{'}_{b2})j_{23} \\ &\quad -r_{a3}r_{b2}j_{23} + (r_{a1} + r^{'}_{a1})(r_{b3} + r^{'}_{b3})j_{13} - r_{a1}r_{b3}j_{13} \\ &\quad +(r_{a2} + r^{'}_{a2})(r_{b3} + r^{'}_{b3})j_{23} - r_{a2}r_{b3}j_{23} \\ &\quad +(r_{a3} + r^{'}_{a3})(r_{b3} + r^{'}_{b3})j_{33} - r_{a3}r_{b3}j_{33}| \\ &\leq |r_{b1}j_{11}|\mathcal{R}_{a1} + |r_{a1}j_{11}|\mathcal{R}_{b1} + |j_{11}|\mathcal{R}_{a1}\mathcal{R}_{b1} + |r_{a2}j_{12}|\mathcal{R}_{b1} \\ &\quad +|r_{b1}j_{12}|\mathcal{R}_{a2} + |j_{12}|\mathcal{R}_{b1}\mathcal{R}_{a2} + |r_{a3}j_{13}|\mathcal{R}_{b1} + |r_{b1}j_{13}|\mathcal{R}_{a3} \\ &\quad +|j_{13}|\mathcal{R}_{b1}\mathcal{R}_{a3} + |r_{a1}j_{12}|\mathcal{R}_{b2} + |r_{b2}j_{12}|\mathcal{R}_{a1} + |j_{12}|\mathcal{R}_{b2}\mathcal{R}_{a1} \\ &\quad +|r_{a2}j_{22}|\mathcal{R}_{b2} + |r_{b2}j_{22}|\mathcal{R}_{a2} + |j_{22}|\mathcal{R}_{a2}\mathcal{R}_{b2} + |r_{a3}j_{23}|\mathcal{R}_{b2} \\ &\quad +|r_{b2}j_{23}|\mathcal{R}_{a3} + |j_{23}|\mathcal{R}_{b2}\mathcal{R}_{a3} + |r_{a1}j_{13}|\mathcal{R}_{b3} + |r_{b3}j_{13}|\mathcal{R}_{a1} \\ &\quad +|j_{13}|\mathcal{R}_{b3}\mathcal{R}_{a1} + |r_{a2}j_{23}|\mathcal{R}_{b3} + |r_{b3}j_{23}|\mathcal{R}_{a2} + |j_{23}|\mathcal{R}_{b3}\mathcal{R}_{a2} \\ &\quad +|r_{a3}j_{33}|\mathcal{R}_{b3} + |r_{b3}j_{33}|\mathcal{R}_{a3} + |j_{33}|\mathcal{R}_{b3}\mathcal{R}_{a3} \\ \Rightarrow \theta_{ab}(\mathbf{J}) &= |r_{b1}j_{11}|\mathcal{R}_{a1} + |r_{a1}j_{11}|\mathcal{R}_{b1} + |j_{11}|\mathcal{R}_{a1}\mathcal{R}_{b1} + |r_{a2}j_{12}|\mathcal{R}_{b1} \\ &\quad +|r_{b1}j_{12}|\mathcal{R}_{a2} + |j_{12}|\mathcal{R}_{b1}\mathcal{R}_{a2} + |r_{a3}j_{13}|\mathcal{R}_{b1} + |r_{b1}j_{13}|\mathcal{R}_{a3} \\ &\quad +|j_{13}|\mathcal{R}_{b1}\mathcal{R}_{a3} + |r_{a1}j_{12}|\mathcal{R}_{b2} + |r_{b2}j_{12}|\mathcal{R}_{a1} + |j_{12}|\mathcal{R}_{b2}\mathcal{R}_{a1} \\ &\quad +|r_{a2}j_{22}|\mathcal{R}_{b2} + |r_{b2}j_{22}|\mathcal{R}_{a2} + |j_{22}|\mathcal{R}_{a2}\mathcal{R}_{b2} + |r_{a3}j_{23}|\mathcal{R}_{b2} \\ &\quad +|r_{b2}j_{23}|\mathcal{R}_{a3} + |j_{23}|\mathcal{R}_{b2}\mathcal{R}_{a3} + |r_{a1}j_{13}|\mathcal{R}_{b3} + |r_{b3}j_{13}|\mathcal{R}_{a1} \\ &\quad +|j_{13}|\mathcal{R}_{b3}\mathcal{R}_{a1} + |r_{a2}j_{23}|\mathcal{R}_{b3} + |r_{b3}j_{23}|\mathcal{R}_{a2} + |j_{23}|\mathcal{R}_{b3}\mathcal{R}_{a2} \\ &\quad +|r_{a3}j_{33}|\mathcal{R}_{b3} + |r_{b3}j_{33}|\mathcal{R}_{a3} + |j_{33}|\mathcal{R}_{b3}\mathcal{R}_{a3} \end{aligned}$$

Now let $\mathbf{M}'$ be a perturbed matrix $\mathbf{M}$, and $\mathcal{M}^*$ be the perturbation such that $\mathbf{M}' = \mathbf{M} + \mathcal{M}^*$. The final goal is to put a bound on the perturbation $\mathcal{M}^*$. We have

$$\mathbf{M} = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & m_{23} \\ m_{13} & m_{23} & m_{33} \end{pmatrix}$$

with

$$\begin{aligned}
m_{11} &= q_{33}(\sum_{i=1}^{n} f_{yi}^2 \mathbf{F}'_i) - 2q_{23}(\sum_{i=1}^{n} f_{yi} f_{zi} \mathbf{F}'_i) + q_{22}(\sum_{i=1}^{n} f_{zi}^2 \mathbf{F}'_i) \\
m_{22} &= q_{11}(\sum_{i=1}^{n} f_{zi}^2 \mathbf{F}'_i) - 2q_{13}(\sum_{i=1}^{n} f_{xi} f_{zi} \mathbf{F}'_i) + q_{33}(\sum_{i=1}^{n} f_{xi}^2 \mathbf{F}'_i) \\
m_{33} &= q_{22}(\sum_{i=1}^{n} f_{xi}^2 \mathbf{F}'_i) - 2q_{12}(\sum_{i=1}^{n} f_{xi} f_{yi} \mathbf{F}'_i) + q_{11}(\sum_{i=1}^{n} f_{yi}^2 \mathbf{F}'_i) \\
m_{12} &= q_{13}(\sum_{i=1}^{n} f_{yi} f_{zi} \mathbf{F}'_i) - q_{33}(\sum_{i=1}^{n} f_{xi} f_{yi} \mathbf{F}'_i) \\
&\quad - q_{12}(\sum_{i=1}^{n} f_{zi} f_{zi} \mathbf{F}'_i) + q_{23}(\sum_{i=1}^{n} f_{xi} f_{zi} \mathbf{F}'_i) \\
m_{13} &= q_{23}(\sum_{i=1}^{n} f_{xi} f_{yi} \mathbf{F}'_i) - q_{13}(\sum_{i=1}^{n} f_{yi} f_{yi} \mathbf{F}'_i) \\
&\quad - q_{22}(\sum_{i=1}^{n} f_{xi} f_{zi} \mathbf{F}'_i) + q_{12}(\sum_{i=1}^{n} f_{yi} f_{zi} \mathbf{F}'_i) \\
m_{23} &= q_{12}(\sum_{i=1}^{n} f_{xi} f_{zi} \mathbf{F}'_i) - q_{11}(\sum_{i=1}^{n} f_{yi} f_{zi} \mathbf{F}'_i) \\
&\quad - q_{23}(\sum_{i=1}^{n} f_{xi} f_{xi} \mathbf{F}'_i) + q_{13}(\sum_{i=1}^{n} f_{xi} f_{yi} \mathbf{F}'_i)
\end{aligned}$$

Similarly, we have $\mathcal{M}^*_{ij}$ denoting the elements of $\mathcal{M}^*$ and $\mathcal{M}_{ij}$ denoting the ones of $\mathcal{M}$. $\mathcal{M}$ denotes the seeked bound on the absolute variation of $\mathbf{M}$. We obtain

$$
\begin{aligned}
|\mathcal{M}^*_{11}| &= |m'_{11} - m_{11}| \\
&= |q'_{33}(\sum_{i=1}^{n} f^2_{yi}\mathbf{F}'_i) - 2q'_{23}(\sum_{i=1}^{n} f_{yi}f_{zi}\mathbf{F}'_i) + q'_{22}(\sum_{i=1}^{n} f^2_{zi}\mathbf{F}'_i) \\
&\quad -q_{33}(\sum_{i=1}^{n} f^2_{yi}\mathbf{F}'_i) + 2q_{23}(\sum_{i=1}^{n} f_{yi}f_{zi}\mathbf{F}'_i) - q_{22}(\sum_{i=1}^{n} f^2_{zi}\mathbf{F}'_i)| \\
&= |q^*_{33}(\sum_{i=1}^{n} f^2_{yi}\mathbf{F}'_i) - 2q^*_{23}(\sum_{i=1}^{n} f_{yi}f_{zi}\mathbf{F}'_i) + q^*_{22}(\sum_{i=1}^{n} f^2_{zi}\mathbf{F}'_i)| \\
&\leq \theta_{33}(\sum_{i=1}^{n} f^2_{yi}\mathbf{F}'_i) + 2\theta_{23}(\sum_{i=1}^{n} f_{yi}f_{zi}\mathbf{F}'_i) + \theta_{22}(\sum_{i=1}^{n} f^2_{zi}\mathbf{F}'_i) \\
\Rightarrow \mathcal{M}_{11} &= \theta_{33}(\sum_{i=1}^{n} f^2_{yi}\mathbf{F}'_i) + 2\theta_{23}(\sum_{i=1}^{n} f_{yi}f_{zi}\mathbf{F}'_i) + \theta_{22}(\sum_{i=1}^{n} f^2_{zi}\mathbf{F}'_i)
\end{aligned}
$$

and analoguous derivations lead to

$$
\begin{aligned}
\mathcal{M}_{22} &= \theta_{11}(\sum_{i=1}^{n} f^2_{zi}\mathbf{F}'_i) + 2\theta_{13}(\sum_{i=1}^{n} f_{xi}f_{zi}\mathbf{F}'_i) + \theta_{33}(\sum_{i=1}^{n} f^2_{xi}\mathbf{F}'_i) \\
\mathcal{M}_{33} &= \theta_{22}(\sum_{i=1}^{n} f^2_{xi}\mathbf{F}'_i) + 2\theta_{12}(\sum_{i=1}^{n} f_{xi}f_{yi}\mathbf{F}'_i) + \theta_{11}(\sum_{i=1}^{n} f^2_{yi}\mathbf{F}'_i) \\
\mathcal{M}_{12} &= \theta_{13}(\sum_{i=1}^{n} f_{yi}f_{zi}\mathbf{F}'_i) + \theta_{33}(\sum_{i=1}^{n} f_{xi}f_{yi}\mathbf{F}'_i) \\
&\quad +\theta_{12}(\sum_{i=1}^{n} f_{zi}f_{zi}\mathbf{F}'_i) + \theta_{23}(\sum_{i=1}^{n} f_{xi}f_{zi}\mathbf{F}'_i) \\
\mathcal{M}_{13} &= \theta_{23}(\sum_{i=1}^{n} f_{xi}f_{yi}\mathbf{F}'_i) + \theta_{13}(\sum_{i=1}^{n} f_{yi}f_{yi}\mathbf{F}'_i) \\
&\quad +\theta_{22}(\sum_{i=1}^{n} f_{xi}f_{zi}\mathbf{F}'_i) + \theta_{12}(\sum_{i=1}^{n} f_{yi}f_{zi}\mathbf{F}'_i) \\
\mathcal{M}_{23} &= \theta_{12}(\sum_{i=1}^{n} f_{xi}f_{zi}\mathbf{F}'_i) + \theta_{11}(\sum_{i=1}^{n} f_{yi}f_{zi}\mathbf{F}'_i) \\
&\quad +\theta_{23}(\sum_{i=1}^{n} f_{xi}f_{xi}\mathbf{F}'_i) + \theta_{13}(\sum_{i=1}^{n} f_{xi}f_{yi}\mathbf{F}'_i)
\end{aligned}
$$

2.3 Bound on the largest Eigenvalue of $\mathrm{M}^{-1/2}\mathcal{M}^*\mathrm{M}^{-1/2}$

As already mentioned in the paper, the bounding of the spectral norm of

$$\mathbf{M}^{-1/2}\mathcal{M}^*\mathbf{M}^{-1/2}$$

requires to bound the absolute value of the roots of the characteristic polynomial of this matrix, which in turn—due to the Lagrangian bound—requires to bound the absolute value of the coefficients a_2, a_1, and a_0 of the characteristic polynomial. The latter is given by det$(\mathbf{M}^{-1/2}\mathcal{M}^*\mathbf{M}^{-1/2} - \mu\mathbf{I}_3) = -\mu^3 + a_2\mu^2 + a_1\mu + a_0$. In order to simlify the notation a bit, let's introduce another variable $\mathbf{S}$ with s_{ij} being the elements of $\mathbf{S}$, and such that $\mathbf{S} = \mathbf{M}^{-1/2}$. The coefficients result to:

$$\begin{aligned} a_2 \;=\;& \mathcal{M}^*_{11}s^2_{11} + \mathcal{M}^*_{11}s^2_{12} + \mathcal{M}^*_{11}s^2_{13} + \mathcal{M}^*_{22}s^2_{12} + \mathcal{M}^*_{22}s^2_{22} + \mathcal{M}^*_{22}s^2_{23} + \mathcal{M}^*_{33}s^2_{13} \\ & + \mathcal{M}^*_{33}s^2_{23} + \mathcal{M}^*_{33}s^2_{33} + 2\mathcal{M}^*_{12}s_{11}s_{12} + 2\mathcal{M}^*_{12}s_{12}s_{22} + 2\mathcal{M}^*_{13}s_{11}s_{13} \\ & + 2\mathcal{M}^*_{12}s_{13}s_{23} + 2\mathcal{M}^*_{13}s_{12}s_{23} + 2\mathcal{M}^*_{23}s_{12}s_{13} + 2\mathcal{M}^*_{13}s_{13}s_{33} \\ & + 2\mathcal{M}^*_{23}s_{22}s_{23} + 2\mathcal{M}^*_{23}s_{23}s_{33} \end{aligned}$$

$$
\begin{aligned}
a_1 \;=\; & \mathcal{M}^{*2}_{12}s^4_{12} + \mathcal{M}^{*2}_{13}s^4_{13} + \mathcal{M}^{*2}_{23}s^4_{23} + \mathcal{M}^{*2}_{12}s^2_{11}s^2_{22} + \mathcal{M}^{*2}_{12}s^2_{11}s^2_{23} \\
& +\mathcal{M}^{*2}_{12}s^2_{12}s^2_{13} + \mathcal{M}^{*2}_{12}s^2_{12}s^2_{23} + \mathcal{M}^{*2}_{12}s^2_{13}s^2_{22} + \mathcal{M}^{*2}_{13}s^2_{11}s^2_{23} + \mathcal{M}^{*2}_{13}s^2_{12}s^2_{13} \\
& +\mathcal{M}^{*2}_{13}s^2_{11}s^2_{33} + \mathcal{M}^{*2}_{13}s^2_{12}s^2_{33} + \mathcal{M}^{*2}_{13}s^2_{13}s^2_{23} + \mathcal{M}^{*2}_{23}s^2_{12}s^2_{23} + \mathcal{M}^{*2}_{23}s^2_{13}s^2_{22} \\
& +\mathcal{M}^{*2}_{23}s^2_{12}s^2_{33} + \mathcal{M}^{*2}_{23}s^2_{13}s^2_{23} + \mathcal{M}^{*2}_{23}s^2_{22}s^2_{33} - \mathcal{M}^{*}_{11}\mathcal{M}^{*}_{22}s^4_{12} - \mathcal{M}^{*}_{11}\mathcal{M}^{*}_{33}s^4_{13} \\
& -\mathcal{M}^{*}_{22}\mathcal{M}^{*}_{33}s^4_{23} - \mathcal{M}^{*}_{11}\mathcal{M}^{*}_{22}s^2_{11}s^2_{22} - \mathcal{M}^{*}_{11}\mathcal{M}^{*}_{22}s^2_{11}s^2_{23} - \mathcal{M}^{*}_{11}\mathcal{M}^{*}_{22}s^2_{12}s^2_{13} \\
& -\mathcal{M}^{*}_{11}\mathcal{M}^{*}_{22}s^2_{12}s^2_{23} - \mathcal{M}^{*}_{11}\mathcal{M}^{*}_{22}s^2_{13}s^2_{22} - \mathcal{M}^{*}_{11}\mathcal{M}^{*}_{33}s^2_{11}s^2_{23} \\
& -\mathcal{M}^{*}_{11}\mathcal{M}^{*}_{33}s^2_{12}s^2_{13} - \mathcal{M}^{*}_{11}\mathcal{M}^{*}_{33}s^2_{11}s^2_{33} - \mathcal{M}^{*}_{11}\mathcal{M}^{*}_{33}s^2_{12}s^2_{33} \\
& -\mathcal{M}^{*}_{11}\mathcal{M}^{*}_{33}s^2_{13}s^2_{23} - \mathcal{M}^{*}_{22}\mathcal{M}^{*}_{33}s^2_{12}s^2_{23} - \mathcal{M}^{*}_{22}\mathcal{M}^{*}_{33}s^2_{13}s^2_{22} \\
& -\mathcal{M}^{*}_{22}\mathcal{M}^{*}_{33}s^2_{12}s^2_{33} - \mathcal{M}^{*}_{22}\mathcal{M}^{*}_{33}s^2_{13}s^2_{23} - \mathcal{M}^{*}_{22}\mathcal{M}^{*}_{33}s^2_{22}s^2_{33} \\
& -2\mathcal{M}^{*2}_{12}s_{11}s^2_{12}s_{22} - 2\mathcal{M}^{*2}_{13}s_{11}s^2_{13}s_{33} - 2\mathcal{M}^{*2}_{23}s_{22}s^2_{23}s_{33} - 2\mathcal{M}^{*}_{11}\mathcal{M}^{*}_{23}s_{12}s^3_{13} \\
& -2\mathcal{M}^{*}_{11}\mathcal{M}^{*}_{23}s^3_{12}s_{13} + 2\mathcal{M}^{*}_{12}\mathcal{M}^{*}_{13}s_{12}s^3_{13} + 2\mathcal{M}^{*}_{12}\mathcal{M}^{*}_{13}s^3_{12}s_{13} \\
& +2\mathcal{M}^{*}_{12}\mathcal{M}^{*}_{23}s_{12}s^3_{23} + 2\mathcal{M}^{*}_{12}\mathcal{M}^{*}_{23}s^3_{12}s_{23} - 2\mathcal{M}^{*}_{13}\mathcal{M}^{*}_{22}s_{12}s^3_{23} \\
& -2\mathcal{M}^{*}_{13}\mathcal{M}^{*}_{22}s^3_{12}s_{23} - 2\mathcal{M}^{*}_{12}\mathcal{M}^{*}_{33}s_{13}s^3_{23} - 2\mathcal{M}^{*}_{12}\mathcal{M}^{*}_{33}s^3_{13}s_{23} \\
& +2\mathcal{M}^{*}_{13}\mathcal{M}^{*}_{23}s_{13}s^3_{23} + 2\mathcal{M}^{*}_{13}\mathcal{M}^{*}_{23}s^3_{13}s_{23} + 2\mathcal{M}^{*}_{11}\mathcal{M}^{*}_{22}s_{11}s^2_{12}s_{22} \\
& +2\mathcal{M}^{*}_{11}\mathcal{M}^{*}_{23}s_{11}s^2_{12}s_{23} - 2\mathcal{M}^{*}_{12}\mathcal{M}^{*}_{13}s_{11}s^2_{12}s_{23} + 2\mathcal{M}^{*}_{11}\mathcal{M}^{*}_{23}s_{11}s^2_{13}s_{23} \\
& -2\mathcal{M}^{*}_{11}\mathcal{M}^{*}_{23}s^2_{11}s_{22}s_{23} - 2\mathcal{M}^{*}_{12}\mathcal{M}^{*}_{13}s_{11}s^2_{13}s_{23} + 2\mathcal{M}^{*}_{12}\mathcal{M}^{*}_{13}s^2_{11}s_{22}s_{23} \\
& +2\mathcal{M}^{*}_{12}\mathcal{M}^{*}_{23}s_{11}s_{13}s^2_{22} - 2\mathcal{M}^{*}_{13}\mathcal{M}^{*}_{22}s_{11}s_{13}s^2_{22} + 2\mathcal{M}^{*}_{11}\mathcal{M}^{*}_{23}s_{12}s_{13}s^2_{23} \\
& -2\mathcal{M}^{*}_{12}\mathcal{M}^{*}_{13}s_{12}s_{13}s^2_{23} + 2\mathcal{M}^{*}_{12}\mathcal{M}^{*}_{23}s_{11}s_{13}s^2_{23} - 2\mathcal{M}^{*}_{12}\mathcal{M}^{*}_{23}s^2_{12}s_{13}s_{22} \\
& -2\mathcal{M}^{*}_{12}\mathcal{M}^{*}_{33}s_{11}s_{12}s^2_{23} - 2\mathcal{M}^{*}_{13}\mathcal{M}^{*}_{22}s_{11}s_{13}s^2_{23} + 2\mathcal{M}^{*}_{13}\mathcal{M}^{*}_{22}s^2_{12}s_{13}s_{22} \\
& +2\mathcal{M}^{*}_{13}\mathcal{M}^{*}_{23}s_{11}s_{12}s^2_{23} - 2\mathcal{M}^{*}_{11}\mathcal{M}^{*}_{23}s^2_{11}s_{23}s_{33} - 2\mathcal{M}^{*}_{11}\mathcal{M}^{*}_{23}s^2_{13}s_{22}s_{23} \\
& +2\mathcal{M}^{*}_{11}\mathcal{M}^{*}_{33}s_{11}s^2_{13}s_{33} + 2\mathcal{M}^{*}_{12}\mathcal{M}^{*}_{13}s^2_{11}s_{23}s_{33} + 2\mathcal{M}^{*}_{12}\mathcal{M}^{*}_{13}s^2_{13}s_{22}s_{23} \\
& -2\mathcal{M}^{*}_{12}\mathcal{M}^{*}_{23}s_{12}s^2_{13}s_{23} - 2\mathcal{M}^{*}_{12}\mathcal{M}^{*}_{33}s_{11}s_{12}s^2_{33} - 2\mathcal{M}^{*}_{12}\mathcal{M}^{*}_{33}s_{12}s^2_{13}s_{22} \\
& +2\mathcal{M}^{*}_{13}\mathcal{M}^{*}_{22}s_{12}s^2_{13}s_{23} + 2\mathcal{M}^{*}_{13}\mathcal{M}^{*}_{23}s_{11}s_{12}s^2_{33} + 2\mathcal{M}^{*}_{13}\mathcal{M}^{*}_{23}s_{12}s^2_{13}s_{22} \\
& -2\mathcal{M}^{*}_{11}\mathcal{M}^{*}_{23}s^2_{12}s_{23}s_{33} + 2\mathcal{M}^{*}_{12}\mathcal{M}^{*}_{13}s^2_{12}s_{23}s_{33} - 2\mathcal{M}^{*}_{12}\mathcal{M}^{*}_{23}s_{13}s_{22}s^2_{23} \\
& +2\mathcal{M}^{*}_{12}\mathcal{M}^{*}_{23}s^2_{12}s_{13}s_{33} + 2\mathcal{M}^{*}_{12}\mathcal{M}^{*}_{33}s^2_{12}s_{13}s_{23} + 2\mathcal{M}^{*}_{13}\mathcal{M}^{*}_{22}s_{13}s_{22}s^2_{23} \\
& -2\mathcal{M}^{*}_{13}\mathcal{M}^{*}_{22}s^2_{12}s_{13}s_{33} - 2\mathcal{M}^{*}_{13}\mathcal{M}^{*}_{23}s^2_{12}s_{13}s_{23} + 2\mathcal{M}^{*}_{12}\mathcal{M}^{*}_{23}s_{13}s^2_{22}s_{33} \\
& -2\mathcal{M}^{*}_{12}\mathcal{M}^{*}_{33}s_{12}s_{22}s^2_{33} + 2\mathcal{M}^{*}_{12}\mathcal{M}^{*}_{33}s_{12}s^2_{13}s_{33} - 2\mathcal{M}^{*}_{13}\mathcal{M}^{*}_{22}s_{13}s^2_{22}s_{33} \\
& +2\mathcal{M}^{*}_{13}\mathcal{M}^{*}_{23}s_{12}s_{22}s^2_{33} - 2\mathcal{M}^{*}_{13}\mathcal{M}^{*}_{23}s_{12}s^2_{13}s_{33} + 2\mathcal{M}^{*}_{12}\mathcal{M}^{*}_{33}s_{12}s^2_{23}s_{33} \\
& -2\mathcal{M}^{*}_{13}\mathcal{M}^{*}_{23}s_{12}s^2_{23}s_{33} + 2\mathcal{M}^{*}_{22}\mathcal{M}^{*}_{33}s_{22}s^2_{23}s_{33} - 2\mathcal{M}^{*2}_{12}s_{11}s_{12}s_{13}s_{23} \\
& -2\mathcal{M}^{*2}_{13}s_{11}s_{12}s_{13}s_{23} - 2\mathcal{M}^{*2}_{12}s_{12}s_{13}s_{22}s_{23} - 2\mathcal{M}^{*2}_{23}s_{12}s_{13}s_{22}s_{23} \\
& -2\mathcal{M}^{*2}_{13}s_{12}s_{13}s_{23}s_{33} - 2\mathcal{M}^{*2}_{23}s_{12}s_{13}s_{23}s_{33} + 2\mathcal{M}^{*}_{11}\mathcal{M}^{*}_{22}s_{11}s_{12}s_{13}s_{23} \\
& +2\mathcal{M}^{*}_{11}\mathcal{M}^{*}_{23}s_{11}s_{12}s_{13}s_{22} - 2\mathcal{M}^{*}_{12}\mathcal{M}^{*}_{13}s_{11}s_{12}s_{13}s_{22} + 2\mathcal{M}^{*}_{11}\mathcal{M}^{*}_{22}s_{12}s_{13}s_{22}s_{23} \\
& +2\mathcal{M}^{*}_{11}\mathcal{M}^{*}_{23}s_{11}s_{12}s_{13}s_{33} + 2\mathcal{M}^{*}_{11}\mathcal{M}^{*}_{33}s_{11}s_{12}s_{13}s_{23} - 2\mathcal{M}^{*}_{12}\mathcal{M}^{*}_{13}s_{11}s_{12}s_{13}s_{33} \\
& -2\mathcal{M}^{*}_{12}\mathcal{M}^{*}_{23}s_{11}s_{12}s_{22}s_{23} + 2\mathcal{M}^{*}_{13}\mathcal{M}^{*}_{22}s_{11}s_{12}s_{22}s_{23} + 2\mathcal{M}^{*}_{11}\mathcal{M}^{*}_{23}s_{12}s_{13}s_{22}s_{33} \\
& -2\mathcal{M}^{*}_{12}\mathcal{M}^{*}_{13}s_{12}s_{13}s_{22}s_{33} - 2\mathcal{M}^{*}_{12}\mathcal{M}^{*}_{23}s_{11}s_{12}s_{23}s_{33} + 2\mathcal{M}^{*}_{12}\mathcal{M}^{*}_{33}s_{11}s_{13}s_{22}s_{23} \\
& +2\mathcal{M}^{*}_{13}\mathcal{M}^{*}_{22}s_{11}s_{12}s_{23}s_{33} - 2\mathcal{M}^{*}_{13}\mathcal{M}^{*}_{23}s_{11}s_{13}s_{22}s_{23} + 2\mathcal{M}^{*}_{11}\mathcal{M}^{*}_{33}s_{12}s_{13}s_{23}s_{33} \\
& -2\mathcal{M}^{*}_{12}\mathcal{M}^{*}_{23}s_{12}s_{22}s_{23}s_{33} + 2\mathcal{M}^{*}_{12}\mathcal{M}^{*}_{33}s_{11}s_{13}s_{23}s_{33} + 2\mathcal{M}^{*}_{13}\mathcal{M}^{*}_{22}s_{12}s_{22}s_{23}s_{33} \\
& -2\mathcal{M}^{*}_{13}\mathcal{M}^{*}_{23}s_{11}s_{13}s_{23}s_{33} + 2\mathcal{M}^{*}_{22}\mathcal{M}^{*}_{33}s_{12}s_{13}s_{22}s_{23} + 2\mathcal{M}^{*}_{12}\mathcal{M}^{*}_{33}s_{13}s_{22}s_{23}s_{33} \\
& -2\mathcal{M}^{*}_{13}\mathcal{M}^{*}_{23}s_{13}s_{22}s_{23}s_{33} + 2\mathcal{M}^{*}_{22}\mathcal{M}^{*}_{33}s_{12}s_{13}s_{23}s_{33}
\end{aligned}
$$

$$
\begin{aligned}
a_0 \;=\; & -\mathcal{M}^*_{11}\mathcal{M}^{*2}_{23}s^2_{11}s^4_{23} - \mathcal{M}^*_{11}\mathcal{M}^{*2}_{23}s^4_{13}s^2_{22} - \mathcal{M}^{*2}_{13}\mathcal{M}^*_{22}s^2_{11}s^4_{23} \\
& -\mathcal{M}^*_{11}\mathcal{M}^{*2}_{23}s^4_{12}s^2_{33} - \mathcal{M}^{*2}_{12}\mathcal{M}^*_{33}s^2_{11}s^4_{23} - \mathcal{M}^{*2}_{13}\mathcal{M}^*_{22}s^4_{13}s^2_{22} \\
& -\mathcal{M}^{*2}_{12}\mathcal{M}^*_{33}s^4_{13}s^2_{22} - \mathcal{M}^{*2}_{13}\mathcal{M}^*_{22}s^4_{12}s^2_{33} - \mathcal{M}^{*2}_{12}\mathcal{M}^*_{33}s^4_{12}s^2_{33} \\
& -\mathcal{M}^*_{11}\mathcal{M}^{*2}_{23}s^2_{11}s^2_{22}s^2_{33} - 4\mathcal{M}^*_{11}\mathcal{M}^{*2}_{23}s^2_{12}s^2_{13}s^2_{23} - \mathcal{M}^{*2}_{13}\mathcal{M}^*_{22}s^2_{11}s^2_{22}s^2_{33} \\
& -4\mathcal{M}^{*2}_{13}\mathcal{M}^*_{22}s^2_{12}s^2_{13}s^2_{23} - \mathcal{M}^{*2}_{12}\mathcal{M}^*_{33}s^2_{11}s^2_{22}s^2_{33} - 4\mathcal{M}^{*2}_{12}\mathcal{M}^*_{33}s^2_{12}s^2_{13}s^2_{23} \\
& +\mathcal{M}^*_{11}\mathcal{M}^*_{22}\mathcal{M}^*_{33}s^2_{11}s^4_{23} + 2\mathcal{M}^*_{12}\mathcal{M}^*_{13}\mathcal{M}^*_{23}s^2_{11}s^4_{23} + \mathcal{M}^*_{11}\mathcal{M}^*_{22}\mathcal{M}^*_{33}s^4_{13}s^2_{22} \\
& +2\mathcal{M}^*_{12}\mathcal{M}^*_{13}\mathcal{M}^*_{23}s^4_{13}s^2_{22} + \mathcal{M}^*_{11}\mathcal{M}^*_{22}\mathcal{M}^*_{33}s^4_{12}s^2_{33} + 2\mathcal{M}^*_{12}\mathcal{M}^*_{13}\mathcal{M}^*_{23}s^4_{12}s^2_{33} \\
& +\mathcal{M}^*_{11}\mathcal{M}^*_{22}\mathcal{M}^*_{33}s^2_{11}s^2_{22}s^2_{33} + 4\mathcal{M}^*_{11}\mathcal{M}^*_{22}\mathcal{M}^*_{33}s^2_{12}s^2_{13}s^2_{23} \\
& +2\mathcal{M}^*_{12}\mathcal{M}^*_{13}\mathcal{M}^*_{23}s^2_{11}s^2_{22}s^2_{33} + 8\mathcal{M}^*_{12}\mathcal{M}^*_{13}\mathcal{M}^*_{23}s^2_{12}s^2_{13}s^2_{23} \\
& +2\mathcal{M}^*_{11}\mathcal{M}^{*2}_{23}s_{11}s^2_{12}s_{22}s^2_{33} - 2\mathcal{M}^*_{11}\mathcal{M}^{*2}_{23}s_{11}s^2_{13}s_{22}s^2_{23} \\
& -2\mathcal{M}^*_{11}\mathcal{M}^{*2}_{23}s_{11}s^2_{12}s^2_{23}s_{33} + 2\mathcal{M}^*_{11}\mathcal{M}^{*2}_{23}s_{11}s^2_{13}s^2_{22}s_{33} \\
& +2\mathcal{M}^{*2}_{13}\mathcal{M}^*_{22}s_{11}s^2_{12}s_{22}s^2_{33} - 2\mathcal{M}^{*2}_{13}\mathcal{M}^*_{22}s_{11}s^2_{13}s_{22}s^2_{23} \\
& +2\mathcal{M}^*_{11}\mathcal{M}^{*2}_{23}s^2_{11}s_{22}s^2_{23}s_{33} - 2\mathcal{M}^*_{11}\mathcal{M}^{*2}_{23}s^2_{12}s^2_{13}s_{22}s_{33} \\
& +2\mathcal{M}^{*2}_{12}\mathcal{M}^*_{33}s_{11}s^2_{12}s_{22}s^2_{33} - 2\mathcal{M}^{*2}_{12}\mathcal{M}^*_{33}s_{11}s^2_{13}s_{22}s^2_{23} \\
& -2\mathcal{M}^{*2}_{13}\mathcal{M}^*_{22}s_{11}s^2_{12}s^2_{23}s_{33} + 2\mathcal{M}^{*2}_{13}\mathcal{M}^*_{22}s_{11}s^2_{13}s^2_{22}s_{33} \\
& -2\mathcal{M}^{*2}_{12}\mathcal{M}^*_{33}s_{11}s^2_{12}s^2_{23}s_{33} + 2\mathcal{M}^{*2}_{12}\mathcal{M}^*_{33}s_{11}s^2_{13}s^2_{22}s_{33} \\
& +2\mathcal{M}^{*2}_{13}\mathcal{M}^*_{22}s^2_{11}s_{22}s^2_{23}s_{33} - 2\mathcal{M}^{*2}_{13}\mathcal{M}^*_{22}s^2_{12}s^2_{13}s_{22}s_{33} \\
& +2\mathcal{M}^{*2}_{12}\mathcal{M}^*_{33}s^2_{11}s_{22}s^2_{23}s_{33} - 2\mathcal{M}^{*2}_{12}\mathcal{M}^*_{33}s^2_{12}s^2_{13}s_{22}s_{33} \\
& +4\mathcal{M}^*_{11}\mathcal{M}^{*2}_{23}s_{11}s_{12}s_{13}s^3_{23} + 4\mathcal{M}^{*2}_{13}\mathcal{M}^*_{22}s_{11}s_{12}s_{13}s^3_{23} \\
& +4\mathcal{M}^*_{11}\mathcal{M}^{*2}_{23}s_{12}s^3_{13}s_{22}s_{23} + 4\mathcal{M}^{*2}_{12}\mathcal{M}^*_{33}s_{11}s_{12}s_{13}s^3_{23} \\
& +4\mathcal{M}^{*2}_{13}\mathcal{M}^*_{22}s_{12}s^3_{13}s_{22}s_{23} + 4\mathcal{M}^*_{11}\mathcal{M}^{*2}_{23}s^3_{12}s_{13}s_{23}s_{33} \\
& +4\mathcal{M}^{*2}_{12}\mathcal{M}^*_{33}s_{12}s^3_{13}s_{22}s_{23} + 4\mathcal{M}^{*2}_{13}\mathcal{M}^*_{22}s^3_{12}s_{13}s_{23}s_{33} \\
& +4\mathcal{M}^{*2}_{12}\mathcal{M}^*_{33}s^3_{12}s_{13}s_{23}s_{33} - 2\mathcal{M}^*_{11}\mathcal{M}^*_{22}\mathcal{M}^*_{33}s_{11}s^2_{12}s_{22}s^2_{33} \\
& +2\mathcal{M}^*_{11}\mathcal{M}^*_{22}\mathcal{M}^*_{33}s_{11}s^2_{13}s_{22}s^2_{23} - 4\mathcal{M}^*_{12}\mathcal{M}^*_{13}\mathcal{M}^*_{23}s_{11}s^2_{12}s_{22}s^2_{33} \\
& +4\mathcal{M}^*_{12}\mathcal{M}^*_{13}\mathcal{M}^*_{23}s_{11}s^2_{13}s_{22}s^2_{23} + 2\mathcal{M}^*_{11}\mathcal{M}^*_{22}\mathcal{M}^*_{33}s_{11}s^2_{12}s^2_{23}s_{33} \\
& -2\mathcal{M}^*_{11}\mathcal{M}^*_{22}\mathcal{M}^*_{33}s_{11}s^2_{13}s^2_{22}s_{33} + 4\mathcal{M}^*_{12}\mathcal{M}^*_{13}\mathcal{M}^*_{23}s_{11}s^2_{12}s^2_{23}s_{33} \\
& -4\mathcal{M}^*_{12}\mathcal{M}^*_{13}\mathcal{M}^*_{23}s_{11}s^2_{13}s^2_{22}s_{33} - 2\mathcal{M}^*_{11}\mathcal{M}^*_{22}\mathcal{M}^*_{33}s^2_{11}s_{22}s^2_{23}s_{33} \\
& +2\mathcal{M}^*_{11}\mathcal{M}^*_{22}\mathcal{M}^*_{33}s^2_{12}s^2_{13}s_{22}s_{33} - 4\mathcal{M}^*_{12}\mathcal{M}^*_{13}\mathcal{M}^*_{23}s^2_{11}s_{22}s^2_{23}s_{33} \\
& +4\mathcal{M}^*_{12}\mathcal{M}^*_{13}\mathcal{M}^*_{23}s^2_{12}s^2_{13}s_{22}s_{33} - 4\mathcal{M}^*_{11}\mathcal{M}^*_{22}\mathcal{M}^*_{33}s_{11}s_{12}s_{13}s^3_{23} \\
& -8\mathcal{M}^*_{12}\mathcal{M}^*_{13}\mathcal{M}^*_{23}s_{11}s_{12}s_{13}s^3_{23} - 4\mathcal{M}^*_{11}\mathcal{M}^*_{22}\mathcal{M}^*_{33}s_{12}s^3_{13}s_{22}s_{23} \\
& -8\mathcal{M}^*_{12}\mathcal{M}^*_{13}\mathcal{M}^*_{23}s_{12}s^3_{13}s_{22}s_{23} - 4\mathcal{M}^*_{11}\mathcal{M}^*_{22}\mathcal{M}^*_{33}s^3_{12}s_{13}s_{23}s_{33} \\
& -8\mathcal{M}^*_{12}\mathcal{M}^*_{13}\mathcal{M}^*_{23}s^3_{12}s_{13}s_{23}s_{33} - 4\mathcal{M}^*_{11}\mathcal{M}^{*2}_{23}s_{11}s_{12}s_{13}s_{22}s_{23}s_{33} \\
& -4\mathcal{M}^{*2}_{13}\mathcal{M}^*_{22}s_{11}s_{12}s_{13}s_{22}s_{23}s_{33} - 4\mathcal{M}^{*2}_{12}\mathcal{M}^*_{33}s_{11}s_{12}s_{13}s_{22}s_{23}s_{33} \\
& +4\mathcal{M}^*_{11}\mathcal{M}^*_{22}\mathcal{M}^*_{33}s_{11}s_{12}s_{13}s_{22}s_{23}s_{33} + 8\mathcal{M}^*_{12}\mathcal{M}^*_{13}\mathcal{M}^*_{23}s_{11}s_{12}s_{13}s_{22}s_{23}s_{33}
\end{aligned}
$$

By turing every minus sign into a plus sign, replacing every s_{ij} by its absolute value $|s_{ij}|$, and every $\mathcal{M}^*_{ij}$ by the corresponding element $\mathcal{M}_{ij}$ of $\mathcal{M}$, these expressions represent the seeked absolute bounds α_2, α_1, and α_0 on the coefficients of the characteristic polynomial. The bound on the eigenvalue variation is finally given by $|\lambda_{i,perturbed} - \lambda_i| \leq \lambda_i(\alpha_2 + \alpha_1^{1/2} + \alpha_0^{1/3})$.